Theorem 1. *If the system input satisfies $R(\psi_{ij})v_j - v_i \neq 0$, all relative states of the Kalman filter converge and are exponentially bounded.*

Proof. The observability determinant (9) is not zero when $R(\psi_{ij})v_j - v_i \neq 0$. Therefore, the system satisfies the nonlinear observability rank condition. According to [1], the corresponding estimator converges exponentially and the estimation error is bounded. The detailed convergence proof is omitted for the weak observable systems as many references have already proved it. □

Theorem 2. *For multiple robots with dynamic estimation model* (2)*, if the control inputs follow the initialization process* (10) *and there is an unobservable condition $R(\psi_{ij})v_j - v_i = 0$, then the estimated relative state of the Kalman filter will converge to an unobservable subspace, i.e.*

$$\lim_{t\to\infty} \hat{X}_{ij}(t) \to \{x, y, \psi | \sqrt{x^2+y^2} = z_{\mathrm{GT}}, \psi = \psi_{\mathrm{GT}}\}, \tag{11}$$

and all states $[x_{ij}, y_{ij}, \psi_{ij}]^T$ drift slowly once they reach the subspace. z_{GT} and ψ_{GT} denote a constant distance measurement and constant relative yaw.

Proof. The derivative of the estimate state $\hat{X}$ can be written as:

$$\dot{\hat{X}}_{ij} = f(\hat{X}_{ij}, U_{ij}) + K(z - h(\hat{X}_{ij})). \tag{12}$$

According to [2], the Kalman gain K and the derivative of the error covariance matrix P can be represented by

$$\begin{aligned} &K = PH^T R^{-1} \\ &\dot{P} = AP + PA^T - PH^T R^{-1} HP + BQB^T. \end{aligned} \tag{13}$$

Based on the definition of the Kalman function, the optimal gain K always satisfies the following equations:

$$\frac{\partial \mathrm{tr(P)}}{\partial K} = 0,\ P = \mathrm{cov(X - \hat{X})} = \mathrm{cov(\tilde{X})}, \tag{14}$$

Therefore, if a **unique equilibrium** space of state error $\tilde{X}$ can be found, the relative estimation under the unobservable condition will converge to that space.

The equilibrium space can be found by setting $\dot{\tilde{X}} = \dot{X}_{ij} - \dot{\hat{X}}_{ij}$ to zero. $\dot{X}_{ij} = [0, 0]^T$ can be derived by combining (2), zero yaw rates assumption, and $R(\psi_{ij})v_j - v_i = 0$. Hence, substitute (12) into $\dot{\tilde{X}} = -\dot{\hat{X}}_{ij} = 0$ which yields

$$\begin{bmatrix} R(\hat{\psi}_{ij})v_j - v_i \\ 0 \end{bmatrix} + K(z - h(\hat{X})) = 0. \tag{15}$$

A two-dimensional time-invariant solution for Eq. 15 is:

$$\begin{cases} \hat{x}_{ij}^2 + \hat{y}_{ij}^2 = z_{\mathrm{GT}}^2, \\ \hat{\psi}_{ij} = \psi_{\mathrm{GT}}. \end{cases} \tag{16}$$

Here we prove that (16) is the unique time-invariant solution by studying all cases. Case 1: $R(\hat{\psi}_{ij})v_j - v_i = 0$ and $K = 0$; Case 2: $R(\hat{\psi}_{ij})v_j - v_i = 0$ and $z - h(\hat{X}) = 0$; Case 3: $R(\hat{\psi}_{ij})v_j - v_i \neq 0$ and $K(z - h(\hat{X})) \neq 0$, but they sum to zero.

Case 1 holds only if $PH^T = 0$ according to (13), which furthermore leads to $\dot{P} = AP + PA^T + BQB^T$. Hence, P is independent of distance measurement z from (3), while H is dependent on z from (5). Since H always varies over time due to measurement noise while P does not, $PH^T = 0$ will be a transient condition. Case 2 corresponds to the time-invariant solution in (16). In case 3, K is time variant as it contains the integration of state variables which are in matrix A, B, and H according to (13). Thus, this solution is also transient. Therefore, (16) is the unique time-invariant equilibrium state space, and the estimated states will converge to the equilibrium space as shown in (11). □

In a formation flight, the reference setpoint is $\bar{p}_{ji}$ for the i^{th} robot in the frame of the j^{th} robot. Thus, the control error of the relative position is

$$e_{ij} = \hat{p}_{ij} - \bar{p}_{ij} = \hat{p}_{ij} + R(\hat{\psi}_{ij})\bar{p}_{ji}, \tag{17}$$

where $\hat{p}_{ij}$ is the relative position estimation. Considering the relative system dynamics $\dot{p}_{ij} = R(\psi_{ij})v_j - v_i - Sr_i p_{ij}$, a dynamic inversion formation control law is proposed as

$$v_i = k_{\text{c}} e_{ij} + R(\hat{\psi}_{ij})v_j - Sr_i \hat{p}_{ij}, \tag{18}$$

where k_{c} denotes the control gain, which leads to $\lim e_{ij} \to 0$. Therefore, the real relative position $p_{ij} \approx \hat{p}_{ij} = -e_{ij} - R(\psi_{\text{GT}})\bar{p}_{ji}$ approximates a constant, and the following holds:

$$\dot{p}_{ij} = 0 = R(\psi_{ij})v_j - v_i. \tag{19}$$

This leads to a zero determinant in (9) such that the stable states of formation control cause an unobservable condition for the relative estimation system.

Theorem 3. *Given the converged state estimation p_{ij} and ψ_{ij}, according to Theorem 2, the invariant $\hat{\psi}_{ij}$ and the estimation drift in Problem* 1. *The estimation error will remain converged and bounded even if the multi-robot system is under unobservable maneuvers such as the formation flight.*

Proof. After the initialization and the formation control, relative states satisfy $p_{ij} = \hat{p}_{ij} = \bar{p}_{ij}$. There are two unobservable cases.

Case 1: Define the estimation drift in Problem 1 as Δp_{ij}. The incorrect relative estimation has the following relationship to the real and reference relative positions:

$$\hat{p}_{ij} = \Delta p_{ij} + p_{ij} \neq \bar{p}_{ij}. \tag{20}$$

Substitute (17) into (18), and consider the zero yaw rate assumption, we can get

$$v_i = k_{\text{c}}(\hat{p}_{ij} - \bar{p}_{ij}) + R(\hat{\psi}_{ij})v_j. \tag{21}$$

In view of (19) and (20), the state will become observable again due to the ensuing control actions:

$$R(\psi_{ij})v_j - v_i = k_{\text{c}}(\bar{p}_{ij} - \hat{p}_{ij}) \neq 0. \tag{22}$$

Hence, based on Theorem 1, the estimated relative position $\hat{p}$ will converge again to the real value p.

Case 2: The system is possibly unobservable when $\hat{p} = \bar{p}$ but $\hat{p} \neq p$, which means the relative estimation is incorrect and the system is unobservable. In this case, the relative position will converge to the subspace (the circle trajectory) according to Theorem 2. However, the measurement noise on v_1 and v_i is omnidirectional, so it has components orthogonal to the equilibrium state, leading to case 1 and hence observability. Moreover, external disturbances and actuation noise will lead to non-zero $R(\psi_{i1})v_1 - v_i$, and hence observability. □

References

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